ON INVERSE VARIATIONAL PRINCIPLES FOR A MEASUREMENT OF SIDE-TROUGH SPILLWAYS

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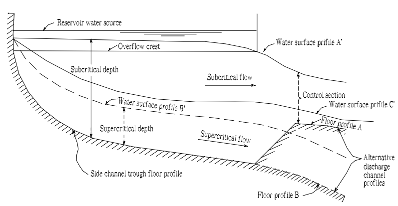
Abstract The hydraulic analysis of side-trough spillways based on the law of conservation of linear momentum is considered. From mathematical point of view the necessary condition that the differential equation provides an extremum of the functional leads to the Euler-Lagrange equation. The inverse variational problem consists of a construction of a functional, using the given differential equation, so that this equation would be an Euler-Lagrange equation for the constructed functional. Two basic approaches have been used for that purpose:a/some special cases when the functional is constructed by means of symmetrical and positive operator and b/ numerical methods for solution of Euler-Lagrange equation. The case b/ leads to "direct" variational principles as Reyleigh-Riesz method, etc. In the case of one side spillway the solution for water depth is used in order to assess the global volume of the trough.

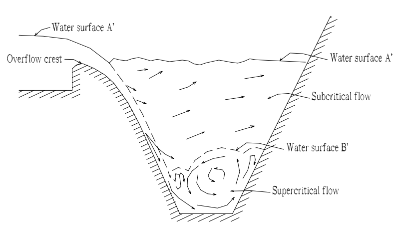
Keywords differential equation, direct variational principle, Euler-Lagrange equation, spillway

Introduction

The side-trough spillways are commonly used in places where the sides are steep and rise to a considerable height above the dam (Figure 1).



(a) Side-Trough profile



(b) Side-Trough section

Figure 1 One side Spillway

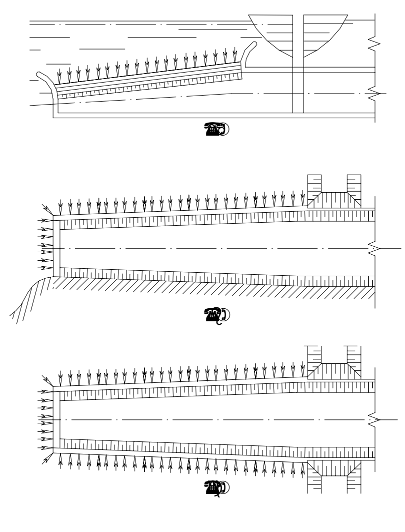


Figure 2 Different Spillways

The side-trough spillways can be constructed on (a) one side, (b) on both sides or (c) on three sides of the trough (Figure 2). In this paper the construction, given in figure 1, is examined closely. The results concerning boundary conditions are discussed as well.

Method

Conservation of linear – momentum

The theory of side-trough spillway is based on the law of conservation of linear momentum. Consider a one-side spillway and co-ordinate system XOY(figure 3).

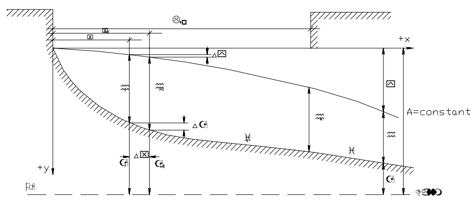


Figure 3 Water flow down the channel

Water flow under steady one dimensional condition is described by Curganov & Dupljak (1982) in the form:

$$d\left(\frac{\alpha v^2}{2g}\right)+\frac{dp}{\gamma}+dz'+i_f dx+\frac{\alpha_0(v-\theta)}{gQ}vdQ=0 \tag{1}$$

The differential form of the flow in side channel spillway is introduced earlier by Hinds (1926). Using the system XOY one can obtain :

$$dy=d\left(\frac{\alpha v^2}{2g}\right)+i_f dx+\alpha_0\frac{(1-\theta/v)}{gw^2}QdQ=0 \tag{2}$$

with α and α_0 kinetic and momentum-energy coefficient , $i_f=\dfrac{Q^2}{w^2c^2R}$ = friction factor

$dz'=i_0 dx[L]$, $i_0=$ bottom slope, $p/\gamma = h$ $[L]$ is the depth of the

section, $y=A-(p/\gamma+z')$, $A=$ constant, w =cross–section area $[L^2]$,

$v=$ velocity, $Q=$ water discharge, θ =velocity component of the additional discharge in

the flow direction , $c=\sqrt{8g/f}$ $[L^{1/2}T^{-1}]$, $R=$ hydraulic radius g =gravity acceleration

After transformations, equation (1) can be rewritten as ordinary differential equation with initial Cauchy condition:

$$\frac{dh}{dx}=\frac{i_0-\frac{Q^2}{K^2}+\frac{\alpha Q^2}{gw^2}\frac{\partial w}{\partial x}-\frac{\alpha_0\left(2-\theta/v\right)dQ/dx}{gw^3}}{1-\frac{\alpha Q^2}{gw^3}B} \tag{3}$$

with $B=$ width of water surface $[L]$, $K=$ equivalent volume modulus $[L^3T^{-1}]$.

The forward problem has to be solved provided that the initial Cauchy condition and geometry of the channel are given. Equation (3) requires an additional data namely the slope of the bottom.

Variational principle for a measurement

We consider equation (2) after neglecting of variables i_f and θ which are of a smaller order. Then the equation can be rewritten in the form:

$$dy=d\left(\frac{\alpha v^2}{2g}\right)+\frac{\alpha_0 v^2}{gQ}dQ \tag{4}$$

Equation (4) leads to the functional

$$I[(v)]=\int_{x_0}^{x_1}H(x)dx \tag{5}$$

where

$$H(x)=h(x)+y(x)=h(x)+\frac{\alpha v^2}{2g}+\frac{\alpha}{g}\int_{x_0}^{x}\frac{v^2(s)q(s)}{Q}ds \tag{6}$$

$$h(x)=\frac{Q(x)}{b(x)\quad v(x)} \tag{7}$$

with $\alpha\approx\alpha_0$, $y(x)=$ co-ordinate shown in Figure 3, $b(x)=$ width of the bottom. If a linear law of water discharge is accepted, the following formulas will be valid:

$$Q(x)=qx \qquad \text{and} \qquad q=Q_{sp}/L_{sp} \tag{8}$$

where $Q(x)$ is a linear function of a specific discharge q that could be calculated by the maximum discharge Q_{sp} and the length of the trough $L_{sp}[L]$.

Having in mind equation (4) the functional (5) can be rewritten as

$$I[v(x)]=\int_{x_0}^{x_1}\left[\frac{Q(x)}{b(x)v(x)}+\frac{\alpha}{2g}v^2(x)+\frac{\alpha}{g}(x_1-x)\frac{v^2(x)q}{Q(x)}\right]dx \tag{9}$$

or $$I[v(x)]=\int_{x_0}^{x_1}G\left[x,v(x),v'(x)\right]dx \tag{10}$$

The Euler-Lagrange equations are:

$$\Omega = \{v = v(x) \in c' \,|\, v(x_0) = A, v(x_1) = B\} \quad G_v - \frac{d}{dx} G_{v'} = 0 \qquad . (11)$$

The values A, B are not determined. So the problem with free boundary arises.

From equation (9) follows that $G_{v'} = 0$ and as a final result one can get:

$$-\frac{Q(x)}{b(x)} + \frac{\alpha}{g} v^3(x) + \frac{2\alpha}{g}(x_1 - x)\frac{q}{Q(x)} v^3(x) = 0 \tag{12}$$

With the assumption that q is a constant the velocity could be determined:

$$v(x) = \left[\frac{g}{\alpha} \quad \frac{Q^2(x)}{\left(Q(x)b(x) + 2(x_1 - x)b(x)q\right)}\right]^{\frac{1}{3}} \tag{13}$$

The velocity according to (13) depends on the geometry of the spillway. For non-prizmatic width of the bottom the following formulas are valid:

$$b(x) = b_0 + tg\beta x \ , \ tg\beta = \frac{b_n - b_0}{L_{sp}} \text{ and } b_n, \ b_0 \text{ are constants} \tag{14}$$

Thus the boundary conditions in the domain of definition Ω are:

$$v(0) = 0, \quad v\left(L_{sp}\right) = \left(\frac{g}{\alpha b_n} qL_{sp}\right)^{\frac{1}{3}} \tag{15}$$

If we consider the cases (b), (c), shown in Figure 2, the equation (13) is modified according to

- both sides of the trough: $Q(x) = q(x) + Q_0, \quad q = \dfrac{Q_{sp}}{L'_{sp}}$

$$v(x) = \left[\frac{g}{\alpha} \quad \frac{\left(Q_{sp}x + Q_0 L'_{sp}\right)^2}{L'_{sp}\left(b_0 + tg\beta x\right)\left(2Q_{sp}L'_{sp} + Q_0 L'_{sp} - Q_{sp}x\right)}\right]^{\frac{1}{3}} \tag{16}$$

where Q_0 and Q_{sp} are the maximum discharges along the short and long side of the spillway respectively ; L'_{sp} is the length of the long side.

- three sides:

$$Q(x) = qx + Q_0, \quad q = \frac{Q_{sp}}{L''_{sp}}$$

$L''_{sp} = 2L'_{sp};\ Q_0$ is a maximum discharge along the short side. The formula for the velocity looks the same way as (16) provided that the long side is symmetrical. The boundary conditions are received from (16) at $x_0 = 0$ and $x_1 = L'_{sp}$ respectively

$$v(0) = \left[\frac{g}{\alpha} \quad \frac{Q_0^2}{b_0\left(2Q_{sp} + Q_0\right)}\right]^{\frac{1}{3}} \qquad v(L_{sp}) = \left[\frac{g}{\alpha} \quad \frac{Q_{sp} + Q_0}{b_n}\right]^{\frac{1}{3}} \tag{17}$$

The final results at the boundaries h_0 and h_n are:

$$h_0 = \left[\frac{\alpha}{g} \frac{2Q_0 Q_{sp} + Q_0^2}{b_0^2} \right]^{\frac{1}{3}} \qquad h_n = \left[\frac{\alpha}{g} \frac{\left(Q_0 + Q_{sp}\right)^2}{b_0^2} \right]^{\frac{1}{3}} \tag{18}$$

The result for one-side construction according to (15) is very close:

$$h_n = \left[\frac{\alpha}{g} q^2 \right]^{\frac{1}{3}} \tag{19}$$

Thus the depth at the end is equal to the critical depth. The same conclusion can be made for overfalling through two or three sides.

Obviously the extremum problem defined by (5) gives minimum lengthwise surface of the trough. If the problem takes place with one-dimensional conditions, the volume of the trough would be minimum also. In this situation the boundary conditions are free and the Euler-Lagrange equation gives a weak extremum of the functional. Using the second variation $\delta^2 I$ the Legendre condition is fulfilled also. From physical point of view the water surface profile A' can be formed by maximum water discharges overfalling above the crest of wear. Then the control section , shown in figure 1(a), must conduct the critical velocity and the lendthwise surface of the trough is minimum. In special cases when the discharge exceeds the maximum one the water surface A' would be above the overflow crest and the weir would work under submerged conditions. This is a risk and emergency management system. There are two interesting components of such system:modelling - forecasting and information systems for management of emergency situations (Cunge and Erlich, 1999). Let consider intermediate profile C' (Figure 1). The depth and the velocity at the control section has to be determined so that the water surface would be under the overflow crest, i.e. submerged conditions are avoided. Now we have isoparametric problem with a given length of the curve of water surface and a distance from the overflow crest so that the weir is protected against submerged conditions. After some transformations the value of constants, parameter λ and boundary conditions can be determined.

Rezults

Numerical methods

Let consider an ordinary or partial differential equation in the following form:

$$Au = f \tag{20}$$

where A = linear differential operator; f = given function

An important special case arises when A is symmetrical and positive defined operator i.e. $(Au, v) = (u, Av)$ and $(Au, \ u) \geq 0$. As it is well known (Gelfand and Fomin, 1962) if equation (20) has a solution, this solution provides minimum of the functional:

$$J(u) = (Au, \ u) - 2(u, \ f) \tag{21}$$

And vice versa: If exists an element that realizes a minimum of the functional (21), this element is a solution of equation (20). We use the inverse part of this statement. After substitution of the formulas for velocities (13) or (16) in equations (6), (7) the expressions for the total depth, respectively for the depth of water could be obtained by numerical simulation.

Simulation

A Riesz numerical method was used for integration of (6) and (7). The results , in case of one side spillway, are shown on table 1.

Table 1 Total volume according minimization and Hinds method.

X	according to (6), (7), (13)		according to Hinds	
[m]	b_0=10 [m]	b_n=20 [m]	b_0=10 [m]	b_n=20 [m]
abscissa	h [m]	H=h+y [m]	h [m]	H=h+y [m]
0	0	0	0	0
10	6.88	7.26	4.51	5.58
20	8.11	9.07	5.91	8.05
30	8.72	10.35	6.74	9.96
40	9.04	11.40	7.29	11.58
50	9.18	12.34	7.66	13.03
60	9.21	13.22	7.92	14.36
70	9.16	14.09	8.10	15.01
80	9.06	14.96	8.22	16.80
90	8.92	15.85	8.29	17.95
100	8.74	16.78	8.34	19.07
110	8.53	17.76	8.36	20.17
117	8.37	18.49	8.36	20.92

Total Volume V_{tr} = 23472 [m^3] V_{tr} = 25290 [m^3]

$\Delta\% = 7.7$

X	according to (6), (7), (13)		according to Hinds	
[m]	b_0=10 [m]	b_n=30 [m]	b_0=10 [m]	b_n=30 [m]
abscissa	h [m[	H=h+y [m]	h [m]	H=h+y [m]
0	0	0	0	0
10	6.54	6.91	3.17	5.65
20	7.41	8.30	4.40	8.33
30	7.71	9.18	5.11	10.27
40	7.77	9.87	5.56	11.81
50	7.71	10.46	5.86	13.11
60	7.58	11.02	6.06	14.24
70	7.41	11.57	6.19	15.26
80	7.22	12.13	6.28	16.19
90	7.01	12.72	6.33	17.06
100	6.79	13.13	6.37	17.87
110	6.55	13.99	6.38	18.64
117	6.39	14.48	6.39	18.87

Total Volume V_{tr} = 26438 [m^3] V_{tr} = 33064 [m^3]

$\Delta\% = 25.06$

The following conclusion has been drown. The results for the volume of trough point to an effect about 25% in comparison with the method of Hinds. This effect grows up with growth of the ratio b_0 / b_n

Table 2 Comparison between forward and inverse problem.

Problem	Input Data	Results
Forward	Cauchy's condition, data for geometry, hydrology, hydraulics, slope i_0	Water surface profile, depth, total depth, volume
Inverse	Free boundary's condition, data for geometry, hydrology, hydraulics	Water surface profile, depth, total depth, volume, slope i_0, two free boundary's conditions

Conclusion

The main idea could be extended to modelling of emergency situation. The method could be used successfully in other identical problems as maximum conductivity of the spillway, conditional extremum according to Lagrange, etc.

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